TOPOLOGICAL PRESSURE AND FRACTAL DIMENSIONS OF COOKIE-CUTTER-LIKE SETS

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arXiv:1203.2724v3 [math.DS] 30 Mar 2012

Abstract. The cookie-cutter-like set is defined as the limit set of a sequence of classical cookie-cutter mappings. For this cookie-cutter-like set first we have determined the topological pressure function, and then by Banach limit we have determined a unique Borel probability measure μ_h with the support the cookie-cutter-like set E. With the topological pressure and the measure μ_h, we have shown that the fractal dimensions such as the Hausdorff dimension, the packing dimension and the box-counting dimension are all equal to the unique zero h of the pressure function. Moreover, we have shown that the h-dimensional Hausdorff measure and the h-dimensional packing measure are finite and positive.

1. Introduction

A basic task in Fractal Geometry is to determine or estimate the various dimensions of fractal sets. Fractal dimensions are introduced to measure the sizes of fractal sets and are used in many different disciplines. Many results on fractal dimensions and measures are obtained, among which the studies on self-similar sets are the most rich and thorough (cf. [H, M, MM, MU, RW, S]). In this paper, we have discussed about the topological pressure, fractal dimensions and measures of a typical fractal known as cookie-cutter-like set.

Let J be a bounded nonempty closed interval in $\mathbb{R}$, and let $J_1, J_2, \cdots, J_N$, $N \geq 2$, be a collection of disjoint closed subintervals of J. Let $f : J_1 \cup \cdots \cup J_N \to J$ be such that each J_j is mapped bijectively onto J. We assume that f has a continuous derivative and is expanding so that $|f'(x)| > 1$ for all $x \in J_1 \cup \cdots \cup J_N$. Let us write

$$E = \{x \in J : f^k(x) \text{ is defined and in } J_1 \cup \cdots \cup J_N \text{ for all } k = 0, 1, 2, \cdots\},$$

where f^k is the kth iterate of f. Thus E is the set of points that remain in $J_1 \cup \cdots \cup J_N$ under the iteration of f. Since $E = \cap_{k=0}^{\infty} f^{-k}(J)$ is the intersection of a decreasing sequence of compact sets, the set E is compact and nonempty. E is invariant under f, in that

$$f(E) = E = f^{-1}(E), \tag{1}$$

since $x \in E$ if and only if $f(x) \in E$. Moreover, E is repeller, in the sense that the points not in E are eventually mapped outside of $J_1 \cup \cdots \cup J_N$ under iteration by f. Let $\varphi_1, \varphi_2, \cdots, \varphi_N$ be the N branches of the inverses of f, i.e., for all $1 \leq j \leq N$, $\varphi_j := (f|_{J_j})^{-1}$, and so $\varphi_j : J \to J$ is such that

$$\varphi_j(x) = f^{-1}(x) \bigcap J_j,$$

and thus $\varphi_1, \varphi_2, \cdots, \varphi_N$ map J bijectively onto $J_1, J_2, \cdots, J_N$ respectively. Since f has a continuous derivative with $|f'(x)| > 1$ on the compact set $J_1 \cup \cdots \cup J_N$, there are numbers $0 < c_{\min} \leq c_{\max} < 1$ such that $1 < c_{\max}^{-1} \leq |f'(x)| \leq c_{\min}^{-1} < \infty$ for all $x \in J_1 \cup \cdots \cup J_N$. It follows that the inverse function φ_j are differentiable with $0 < c_{\min} \leq |\varphi_j'(x)| \leq c_{\max} < 1$ for all $x \in J$ and for all $1 \leq j \leq N$. By the mean value theorem, for $1 \leq j \leq N$, we have

$$c_{\min}|x - y| \leq |\varphi_j(x) - \varphi_j(y)| \leq c_{\max}|x - y|$$

2000 *Mathematics Subject Classification.* 28A80, 28A78.

Key words and phrases. Cookie-cutter-like set, topological pressure, Hausdorff measure, Hausdorff dimension, packing measure, packing dimension, box-counting dimension.

for $x, y \in J$. By (1) the repeller E of f satisfies

$$E = \bigcup_{j=1}^{N} \varphi_j(E).$$

Since each φ_j is a contraction on J, using the fundamental IFS property (cf. [F2, Theorem 2.6]), the repeller E of f is the attractor or the invariant set of the set of contraction mappings $\{\varphi_1, \varphi_2, \cdots, \varphi_N\}$. We say that the collection of contraction mappings $\{\varphi_1, \varphi_2, \cdots, \varphi_N\}$ satisfies the open set condition (OSC) if there exists a bounded nonempty open set $U \subset \mathbb{R}$ such that $\bigcup_{j=1}^{N} \varphi_j(U) \subset U$ with the union disjoint. A dynamical system $f : J_1 \cup \cdots \cup J_N \to J$ of this form is called a cookie-cutter mapping, the equivalent iterated function system $\{\varphi_1, \varphi_2, \cdots, \varphi_N\}$ on J is termed a cookie-cutter system and the set E is called a cookie-cutter set. In general the mappings $\varphi_1, \varphi_2, \cdots, \varphi_N$ are not similarity transformations and E is a 'distorted' Cantor set, which nevertheless is 'approximately self-similar'. If E is a cookie-cutter set with $\dim_H(E) = s$, then it is known that $\underline{\dim}_B E = \overline{\dim}_B E = \dim_P E = s$ and $0 < \mathcal{H}^s(E) < \infty$. For details about it one could see [F2, B]. In this paper, we have considered a sequence $\{f_k\}_{k=1}^{\infty}$ of cookie-cutter mappings, and call the corresponding repeller E the cookie-cutter-like set.

We denote by $\mathcal{H}^s(E)$, $\mathcal{P}^s(E)$, $\dim_H E$, $\dim_P E$, $\underline{\dim}_B E$ and $\overline{\dim}_B E$ the s-dimensional Hausdorff measure, the s-dimensional packing measure, the Hausdorff dimension, the packing dimension, the lower box-counting and the upper box-counting dimension of the set E respectively. In this paper, for the cookie-cutter-like set E we have defined the topological pressure function $P(t)$, and showed that there exists a unique $h \in (0, +\infty)$ such that $P(h) = 0$, and then using Banach limit we have shown that there exists a unique Borel probability measure μ_h supported by E. Using the consequence of the topological pressure and the probability measure, we have shown that

$$\dim_H(E) = \dim_P(E) = \underline{\dim}_B(E) = \overline{\dim}_B(E) = h, \text{ and } 0 < \mathcal{H}^h(E) \leq \mathcal{P}^h(E) < \infty.$$

The result in this paper is a nonlinear extension, as well as a generalization of the classical result about self-similar sets given by the following theorem (cf. [H]):

Theorem: Let E be the limit set generated by a finite system $\{S_1, S_2, \cdots, S_N\}$ of self-similar mappings satisfying the open set condition, where c_n are the similarity ratios of S_n. Then

$$\dim_H(E) = \dim_P(E) = \underline{\dim}_B(E) = \overline{\dim}_B(E) = s, \text{ and } 0 < \mathcal{H}^s(E) \leq \mathcal{P}^s(E) < \infty,$$

where s is the unique positive real number given by $\sum_{j=1}^{N} c_j^s = 1$.

2. Basic results, cookie-cutter-like sets and the topological pressure

In this section, first we adopt the following definitions and notations, which can be found in [F1, F2].

Let E be a nonempty bounded subset of $\mathbb{R}^n$ where $n \geq 1$, and $s \geq 0$. We denote by $\mathcal{H}^s(E)$, $\mathcal{P}^s(E)$, $\dim_H E$, $\dim_P E$, $\underline{\dim}_B E$ and $\overline{\dim}_B E$ the s-dimensional Hausdorff measure, the s-dimensional packing measure, the Hausdorff dimension, the packing dimension, the lower box-counting and the upper box-counting dimension of the set E respectively. There are some basic inequalities between these dimensions:

$$\dim_H E \leq \dim_P E \leq \overline{\dim}_B E, \text{ and } \dim_H E \leq \underline{\dim}_B E \leq \overline{\dim}_B E. \tag{2}$$

Moreover, it is well-known that $\mathcal{H}^s(E) \leq \mathcal{P}^s(E)$. Let $s \geq 0$, and let $\mathcal{U} = \{U_i\}$ be a countable collection of sets of $\mathbb{R}^n$. We define

$$\|\mathcal{U}\|^s := \sum_{U_i \in \mathcal{U}} |U_i|^s,$$

where $|A|$ denotes the diameter of a set A.

2.1. **Cookie-cutter-like set:** Let J be a bounded nonempty closed interval in $\mathbb{R}$. For simplicity we take the diameter of the set J to be one. A mapping f is called a cookie-cutter, if there exists a finite collection of disjoint closed intervals $J_1, J_2, \cdots, J_N \subset J$, such that

$(C1)$ f is defined in a neighborhood of each J_j, $1 \le j \le N$, the restriction of f to each initial interval J_j is 1-1 and onto, the corresponding branch inverse is denoted by $\varphi_j := (f|_{J_j})^{-1} : J \to J_j$, i.e., $\varphi_j(x) = f^{-1}(x) \cap J_j$ for all $1 \le j \le N$;

$(C2)$ f is differentiable with Hölder continuous derivative f', i.e., there exist constants $c_f > 0$ and $\gamma_f \in (0,1]$ such that for $x, y \in J_j$, $1 \le j \le N$,

$$|f'(x) - f'(y)| \le c_f |x-y|^{\gamma_f};$$

$(C3)$ f is boundedly expanding in the sense that there exist constants b_f and B_f such that

$$1 < b_f := \inf_x \{|f'(x)|\} \le \sup_x \{|f'(x)|\} := B_f < +\infty.$$

$[\bigcup_{j=1}^N J_j; c_f, \gamma_f, b_f, B_f]$ is called the defining data of the cookie-cutter mapping f.

Let $N \ge 2$ be fixed, and consider a sequence of cookie-cutter mappings $\{f_k\}_{k\ge1}$ with defining data $[\bigcup_{j=1}^N J_{k,j}; c_k, \gamma_k, b_k, B_k]$. Let us write $\varphi_{k,j} := \left(f_k|_{J_{k,j}}\right)^{-1}$ to denote the corresponding branch inverse of f_k, where $k \ge 1$ and $1 \le j \le N$. We always assume that

$$1 < \inf\{b_k\} \le \sup\{B_k\} < \infty,\ 0 < \inf\{\gamma_k\} \le \sup\{\gamma_k\} \le 1 \text{ and } 0 < \inf\{c_k\} \le \sup\{c_k\} < \infty.$$

Let Ω_0 be the empty set. For $n \ge 1$, define

$$\Omega_n = \{1, 2, \cdots, N\}^n,\ \Omega_\infty = \lim_{n\to\infty} \Omega_n \text{ and } \Omega = \bigcup_{k=0}^\infty \Omega_k.$$

Elements of Ω are called words. For any $\sigma \in \Omega$ if $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_n) \in \Omega_n$, we write $\sigma^- = (\sigma_1, \sigma_2, \cdots, \sigma_{n-1})$ to denote the word obtained by deleting the last letter of σ, $|\sigma| = n$ to denote the length of σ, and $\sigma|_k := (\sigma_1, \sigma_2, \cdots, \sigma_k)$, $k \le n$, to denote the truncation of σ to the length k. For any two words $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_k)$ and $\tau = (\tau_1, \tau_2, \cdots, \tau_m)$, we write $\sigma\tau = \sigma * \tau = (\sigma_1, \cdots, \sigma_k, \tau_1, \cdots, \tau_m)$ to denote the juxtaposition of $\sigma, \tau \in \Omega$. A word of length zero is called the empty word and is denoted by $\emptyset$. For $\sigma \in \Omega$ and $\tau \in \Omega \cup \Omega_\infty$ we say τ is an extension of σ, written as $\sigma \prec \tau$, if $\tau|_{|\sigma|} = \sigma$. For $\sigma \in \Omega_k$, the cylinder set $C(\sigma)$ is defined as $C(\sigma) = \{\tau \in \Omega_\infty : \tau|_k = \sigma\}$. For $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_n) \in \Omega_n$, let us write $\varphi_\sigma = \varphi_{1,\sigma_1} \circ \cdots \circ \varphi_{n,\sigma_n}$, and define the rank-$n$ basic interval corresponding to σ by

$$J_\sigma = J_{(\sigma_1, \sigma_2, \cdots, \sigma_n)} = \varphi_\sigma(J),$$

where $1 \le \sigma_k \le N$, $1 \le k \le n$. It is easy to see that the set of basic intervals $\{J_\sigma : \sigma \in \Omega\}$ has the following net properties:

(i) $J_{\sigma * j} \subset J_\sigma$ for each $\sigma \in \Omega_n$ and $1 \le j \le N$ for all $n \ge 1$;

(ii) $J_\sigma \bigcap J_\tau = \emptyset$, if $\sigma, \tau \in \Omega_n$ for all $n \ge 1$ and $\sigma \ne \tau$.

Let $b = \inf\{b_k\}$ and $B = \sup\{B_k\}$. Then by our assumption, $1 < b \le B < \infty$. Moreover by $(C2)$, as $\varphi_{k,j}$ is a corresponding branch inverse of f_k, where $k \ge 1$ and $1 \le j \le N$, for all $x \in J$, we have $|f_k'(\varphi_{k,j}(x))| \left|\varphi_{k,j}'(x)\right| = 1$, and so

$$B^{-1} \le \left|\varphi_{k,j}'(x)\right| \le b^{-1}. \tag{3}$$

Now let

$$E = \bigcap_{n=0}^\infty \bigcup_{\sigma \in \Omega_n} J_\sigma.$$

Choose x, y to be the end points of J, and then $\varphi_\sigma(x), \varphi_\sigma(y)$ are the end points of J_σ for each $\sigma \in \Omega$, and so by mean value theorem,

$$|J_\sigma| = |\varphi_\sigma(x) - \varphi_\sigma(y)| = |\varphi'_\sigma(w)||x - y| = |\varphi'_\sigma(w)|,$$

for some $w \in J_\sigma$. Thus $B^{-n} \leq |J_\sigma| \leq b^{-n}$ for any $\sigma \in \Omega_n$, and so the diameter $|J_\sigma| \to 0$ as $|\sigma| \to \infty$. Since given $\sigma = (\sigma_i)_{i=1}^\infty \in \Omega_\infty$ the diameters of the compact sets $J_{\sigma|_k}$, $k \geq 1$, converge to zero and since they form a descending family, the set

$$\bigcap_{k=0}^{\infty} J_{\sigma|_k}$$

is a singleton and therefore, if we denote its element by $\pi(\sigma)$, this defines the coding map $\pi : \Omega_\infty \to J$, and so it follows that

$$E = \pi(\Omega_\infty) = \bigcup_{\sigma \in \Omega_\infty} \bigcap_{k=0}^{\infty} J_{\sigma|_k}.$$

Moreover, $\pi(C(\sigma)) = E \cap J_\sigma$ for $\sigma \in \Omega$. With the net properties it follows that E is a perfect, nowhere dense and totally disconnected subset of J. The set E is called the *cookie-cutter-like* (CC-like) set generated by the cookie-cutter sequence $\{f_k\}_{k=1}^\infty$.

Remark 2.2. For each $k \geq 1$, take $f_k = f$, where f is a cookie-cutter mapping, then E is the classical cookie-cutter set, it is the unique invariant set of f, i.e., $E = f^{-1}(E)$.

Definition 2.3. *Let E be a CC-like set and $r > 0$. The family of basic intervals $\mathcal{U}_r = \{J_\sigma : |J_\sigma| \leq r < |J_{\sigma^-}|\} \subset \{J_\sigma : \sigma \in \Omega\}$ is called the r-Moran covering of E provided it is a covering of E, i.e., $E \subseteq \bigcup_{J_\sigma \in \mathcal{U}_r} J_\sigma$.*

From the definition it follows that the elements of a Moran covering are disjoint, have almost equal sizes, and are often of different ranks.

We need the following lemma that appeared in [MRW]. For completeness, the proof is given here.

Lemma 2.4. *(cf.* [MRW, Lemma 2.1]*) (**Bounded variation principle**) There exists a constant $1 < \xi < +\infty$ such that for each $n \geq 1$, $\sigma \in \Omega_n$, and $x, y \in J_\sigma$, we have*

$$\xi^{-1} \leq \frac{|F'_n(x)|}{|F'_n(y)|} \leq \xi$$

where $F_n(x) = f_n \circ f_{n-1} \circ \cdots \circ f_1(x)$.

Proof. Fix $n \geq 1$, $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_n) \in \Omega_n$ and $x, y \in J_\sigma$. Notice that for each $k \leq n$, F_{k-1} maps J_σ diffeomorphically to the set $\varphi_{k,\sigma_k} \circ \varphi_{k+1,\sigma_{k+1}} \circ \cdots \circ \varphi_{n,\sigma_n}(J)$, and so

$$|F_{k-1}(x) - F_{k-1}(y)| \leq \operatorname{diam}\left(\varphi_{k,\sigma_k} \circ \varphi_{k+1,\sigma_{k+1}} \circ \cdots \circ \varphi_{n,\sigma_n}(J)\right) = |\varphi_{k,\sigma_k} \circ \varphi_{k+1,\sigma_{k+1}} \circ \cdots \circ \varphi_{n,\sigma_n}(J)|.$$

By mean value theorem,

$$\begin{aligned}
&|\varphi_{k,\sigma_k} \circ \varphi_{k+1,\sigma_{k+1}} \circ \cdots \circ \varphi_{n,\sigma_n}(J)| \\
&= \sup_{x,y \in J} \left|\varphi_{k,\sigma_k}\left(\varphi_{k+1,\sigma_{k+1}} \circ \cdots \circ \varphi_{n,\sigma_n}(x)\right) - \varphi_{k,\sigma_k}\left(\varphi_{k+1,\sigma_{k+1}} \circ \cdots \circ \varphi_{n,\sigma_n}(y)\right)\right| \\
&\leq b^{-1} \left|\varphi_{k+1,\sigma_{k+1}} \circ \cdots \circ \varphi_{n,\sigma_n}(J)\right|.
\end{aligned}$$

Thus proceeding inductively,

$$\left|\varphi_{k,\sigma_k} \circ \varphi_{k+1,\sigma_{k+1}} \circ \cdots \circ \varphi_{n,\sigma_n}(J)\right| \leq b^{-(n-k+1)}.$$

Then, Hölder continuity of f'_k gives

$$|f'_k(F_{k-1}(x)) - f'_k(F_{k-1}(y))| \leq c\,|F_{k-1}(x) - F_{k-1}(y)|^\gamma \leq cb^{-(n-k+1)\gamma},$$

and so by mean value theorem and the assumption $|f_k'| > 1$, we have

$$\begin{aligned}&\Big|\log|f_k'(F_{k-1}(x))| - \log|f_k'(F_{k-1}(y))|\Big|\\ &\le \Big||f_k'(F_{k-1}(x))| - |f_k'(F_{k-1}(y))|\Big|\\ &\le cb^{-(n-k+1)\gamma}.\end{aligned}$$

Therefore, by the above inequality and the chain rule,

$$\begin{aligned}&\Big|\log|F_n'(x)| - \log|F_n'(y)|\Big|\\ &= \Big|\sum_{k=1}^{n}\log|f_k'(F_{k-1}(x))| - \sum_{k=1}^{n}\log|f_k'(F_{k-1}(y))|\Big|\\ &\le \sum_{k=1}^{n}\Big|\log|f_k'(F_{k-1}(x))| - \log|f_k'(F_{k-1}(y))|\Big|\\ &\le \sum_{k=1}^{n} cb^{-(n-k+1)\gamma} \le \frac{cb^{-\gamma}}{1-b^{-\gamma}}.\end{aligned}$$

Take $\xi = \exp\left\{\frac{c}{b^\gamma - 1}\right\}$. Since $\frac{c}{b^\gamma-1} > 0$, we have $1 < \xi < +\infty$, and thus the proposition follows. □

Let us now prove the following proposition.

Proposition 2.5. . (**Bounded distortion principle**) For any $n \ge 1$, $\sigma \in \Omega_n$, $x \in J_\sigma$, we have

$$\xi^{-1} \le |F_n'(x)| \cdot |J_\sigma| \le \xi,$$

where $F_n(x) = f_n \circ f_{n-1} \circ \cdots \circ f_1(x)$. Moreover, for each $1 \le j \le N$, we get $|J_{\sigma * j}| \ge \xi^{-1}B^{-1}|J_\sigma|$, where ξ is the constant of Lemma 2.4.

Proof. Note that for $\sigma \in \Omega_n$, $F_n : J_\sigma \to J$ is a differentiable bijection. So by mean value theorem, if $y, z \in J_\sigma$, there exists $w \in J_\sigma$ such that

$$F_n(y) - F_n(z) = F_n'(w)(y - z).$$

Choose y, z to be the end points of J_σ, and then $F_n(y), F_n(z)$ are the end points of J, and so

$$|J| = |F_n'(w)| \cdot |J_\sigma|, \text{ i.e. } |F_n'(w)| \cdot |J_\sigma| = 1. \tag{4}$$

Hence, using bounded variation principle, we have

$$\xi^{-1} \le |F_n'(x)| \cdot |J_\sigma| \le \xi \tag{5}$$

for all $x \in J_\sigma$. Thus for $1 \le j \le N$, using (5), we have

$$\xi^{-1} \le |F_{n+1}'(x)| \cdot |J_{\sigma*j}| = |f_n'(F_n(x))| \cdot |F_n'(x)| \cdot |J_{\sigma*j}| \le B|F_n'(x)| \cdot |J_{\sigma*j}|.$$

Then (4) implies

$$|J_{\sigma*j}| \ge \xi^{-1}B^{-1}|J_\sigma|,$$

which completes the proof of the proposition. □

Let us now prove the following proposition.

Proposition 2.6. Let $\sigma \in \Omega_n$, $n \ge 1$, $x, y \in J$ and ξ be the constant of Lemma 2.4. Then,

$$\xi^{-1}|\varphi_\sigma'(y)| \le |\varphi_\sigma'(x)| \le \xi|\varphi_\sigma'(y)|.$$

Proof. For $\sigma \in \Omega_n$, $n \geq 1$, and $x \in J$, we know $F_n(\varphi_\sigma(x)) = x$. Thus

$$|F_n'(\varphi_\sigma(x))| \cdot |\varphi_\sigma'(x)| = 1.$$

Again for all $x \in J$, $\varphi_\sigma(x) \in J_\sigma$. Hence, Lemma 2.4 yields

$$\xi^{-1}|\varphi_\sigma'(y)| \leq |\varphi_\sigma'(x)| \leq \xi|\varphi_\sigma'(y)|,$$

and thus the proposition is obtained. □

For any $\sigma \in \Omega$, let us write $\|\varphi_\sigma\| = \sup_{x\in J} |\varphi_\sigma(x)|$. From the above proposition the following lemma easily follows.

Lemma 2.7. *Let $\sigma, \tau \in \Omega$. Then*

$$\xi^{-1}\|\varphi_\sigma'\|\|\varphi_\tau'\| \leq \|\varphi_{\sigma\tau}'\| \leq \|\varphi_\sigma'\|\|\varphi_\tau'\|.$$

2.8. **Topological pressure:** For $t \in \mathbb{R}$ and $n \geq 1$, let us write $Z_n(t) = \sum_{\sigma\in\Omega_n} \|\varphi_\sigma'\|^t$. Then for $n, p \geq 1$,

$$Z_{n+p}(t) = \sum_{\sigma\in\Omega_n} \sum_{\tau\in\in\Omega_p} \|\varphi_{\sigma\tau}'\|^t.$$

By Lemma 2.7, if $t \geq 0$ then

$$Z_{n+p}(t) \leq Z_n(t)Z_p(t),$$

and if $t < 0$ then

$$Z_{n+p}(t) \leq \xi^{-t}Z_n(t)Z_p(t).$$

Hence by the standard theory of sub-additive sequences, $\lim_{k\to\infty} \frac{1}{k} \log Z_k(t)$ exists (cf. [F2, Corollary 1.2]). Let us denote it by $P(t)$, i.e.,

$$P(t) = \lim_{k\to\infty} \frac{1}{k} \log \sum_{\sigma\in\Omega_k} \|\varphi_\sigma'\|^t. \tag{6}$$

The above function $P(t)$ is called the *topological pressure* of the CC-like set E. Lemma 2.9 and Lemma 2.10 give some properties of the function $P(t)$.

Lemma 2.9. *The function $P(t)$ is strictly decreasing, convex and hence continuous on $\mathbb{R}$.*

Proof. To prove that $P(t)$ is strictly decreasing let $\delta > 0$. Then by Lemma 2.7 and Inequality (3),

$$\begin{aligned} P(t+\delta) &= \lim_{k\to\infty} \frac{1}{k} \log \sum_{\sigma\in\Omega_k} \|\varphi_\sigma'\|^{t+\delta} \leq \lim_{k\to\infty} \frac{1}{k} \log \sum_{\sigma\in\Omega_k} \|\varphi_\sigma'\|^t b^{-k\delta} \\ &= P(q,t) - \delta \log b < P(q,t), \end{aligned}$$

i.e., $P(t)$ is strictly decreasing. For $t_1, t_2 \in \mathbb{R}$ and $a_1, a_2 > 0$ with $a_1 + a_2 = 1$, using Hölder's inequality, we have

$$\begin{aligned} P(a_1t_1 + a_2t_2) &= \lim_{n\to\infty} \frac{1}{k} \log \sum_{\sigma\in\Omega_k} \|\varphi_\sigma'\|^{a_1t_1+a_2t_2} = \lim_{k\to\infty} \frac{1}{k} \log \sum_{\sigma\in\Omega_k} \Big(\|\varphi_\sigma'\|^{t_1}\Big)^{a_1} \Big(\|\varphi_\sigma'\|^{t_2}\Big)^{a_2} \\ &\leq \lim_{k\to\infty} \frac{1}{k} \log \Big(\sum_{\sigma\in\Omega_k} \|\varphi_\sigma'\|^{t_1}\Big)^{a_1} \Big(\sum_{\sigma\in\Omega_k} \|\varphi_\sigma'\|^{t_2}\Big)^{a_2} \\ &= a_1P(t_1) + a_2P(t_2), \end{aligned}$$

i.e., $P(t)$ is convex and hence continuous on $\mathbb{R}$. □

Let us now prove the following lemma.

Lemma 2.10. *There exists a unique $h \in (0, +\infty)$ such that $P(h) = 0$.*

Proof. By Lemma 2.9, the function $P(t)$ is strictly decreasing and continuous on $\mathbb{R}$, and so there exists a unique $h \in \mathbb{R}$ such that $P(h) = 0$. Note that

$$P(0) = \lim_{k\to\infty} \frac{1}{k} \log \sum_{\sigma\in\Omega_k} 1 = \lim_{k\to\infty} \frac{1}{k} \log N^k = \log N \geq \log 2 > 0.$$

In order to conclude the proof it therefore suffices to show that $\lim_{t\to+\infty} P(t) = -\infty$. For $t > 0$,

$$P(t) = \lim_{k\to\infty} \frac{1}{k} \log \sum_{\sigma\in\Omega_k} \|\varphi'_\sigma\|^t \leq \lim_{k\to\infty} \frac{1}{k} \log \sum_{\sigma\in\Omega_k} b^{-kt} = \lim_{k\to\infty} \frac{1}{k} \log N^k - t\log b = \log N - t\log b < 0.$$

Since $b > 1$, it follows that $\lim_{t\to+\infty} P(t) = -\infty$, and hence the lemma follows. □

In the next section, we state and prove the main result of the paper.

3. Main result

The relationship between the unique zero h of the pressure function $P(t)$ and the fractal dimensions of the cookie-cutter-like set E is given by the following theorem. Moreover, it shows that the h-dimensional Hausdorff measure and the h-dimensional packing measure are finite and positive.

Theorem 3.1. *Let E be the cookie-cutter-like set associated with the family $\{f_k\}_{k=1}^\infty$ of cookie-cutter mappings, and $h \in (0, +\infty)$ be such that $P(h) = 0$. Then*

$$dim_H(E) = dim_P(E) = \underline{dim}_B(E) = \overline{dim}_B(E) = h,$$

and

$$0 < \mathcal{H}^h(E) \leq \mathcal{P}^h(E) < \infty.$$

We need the following lemma.

Lemma 3.2. *Let $n \geq 1$, $\sigma \in \Omega_n$ and $x \in J$. Then*

$$\xi^{-1}|J_\sigma| \leq |\varphi'_\sigma(x)| \leq \xi|J_\sigma|,$$

where ξ is the constant of Lemma 2.4.

Proof. Let $x \in J$, and then $\varphi_\sigma(x) \in J_\sigma$, where $\sigma \in \Omega_n$ and $n \geq 1$. Moreover, $F_n(\varphi_\sigma(x)) = x$, and so $|F'_n(\varphi_\sigma(x))| \cdot |\varphi'_\sigma(x)| = 1$ where $F_n(x) = f_n \circ f_{n-1} \circ \cdots \circ f_1(x)$. Now use Proposition 2.5 to obtain the lemma. □

Let us now prove the following lemma.

Lemma 3.3. *Let $\sigma, \tau \in \Omega$. Then*

$$\xi^{-3}|J_\sigma||J_\tau| \leq |J_{\sigma\tau}| \leq \xi^3|J_\sigma||J_\tau|,$$

where ξ is the constant of Lemma 2.4.

Proof. For $\sigma, \tau \in \Omega$, we have $|\varphi'_{\sigma\tau}(x)| = |\varphi'_\sigma(y)||\varphi'_\tau(x)|$ where $y = \varphi_\tau(x)$ for $x \in J$. Again Proposition 2.6 gives that for any $x, y \in J$

$$\xi^{-1}|\varphi'_\sigma(y)| \leq |\varphi'_\sigma(x)| \leq \xi|\varphi'_\sigma(y)|.$$

Hence, Lemma 3.2 implies

$$\xi^{-3}|J_\sigma||J_\tau| \leq \xi^{-1}|\varphi'_\sigma(y)||\varphi'_\tau(x)| = \xi^{-1}|\varphi'_{\sigma\tau}(x)| \leq |J_{\sigma\tau}| \leq \xi|\varphi'_{\sigma\tau}(x)| \leq \xi^3|J_\sigma||J_\tau|,$$

and thus the lemma is obtained. □

Note 3.4. Lemma 3.2 implies

$$\xi^{-1}|J_\sigma| \leq \sup_{x\in J} |\varphi'_\sigma(x)| = \|\varphi'_\sigma\| \leq \xi |J_\sigma|,$$

and so the topological pressure $P(t)$ can be written as follows:

$$P(t) = \lim_{k\to\infty} \frac{1}{k} \log \sum_{\sigma\in\Omega_k} |J_\sigma|^t.$$

Let us now prove the following proposition, which plays a vital role in the paper.

Proposition 3.5. Let $h \in (0, +\infty)$ be unique such that $P(h) = 0$, and let s_* and s^* be any two arbitrary real numbers with $0 < s_* < h < s^*$. Then for all $n \geq 1$,

$$\xi^{-3s_*} < \sum_{\sigma\in\Omega_n} |J_\sigma|^{s_*} \text{ and } \sum_{\sigma\in\Omega_n} |J_\sigma|^{s^*} < \xi^{3s^*},$$

where ξ is the constant of Lemma 2.4.

Proof. Let $s_* < h$. As the pressure function $P(t)$ is strictly decreasing, $P(s_*) > P(h) = 0$. Then for any positive integer n, by Lemma 3.3, we have

$$0 < P(s_*) = \lim_{p\to\infty} \frac{1}{np} \log \sum_{\omega\in\Omega_{np}} |J_\omega|^{s_*} \leq \lim_{p\to\infty} \frac{1}{np} \log \xi^{3(p-1)s_*} \left(\sum_{\sigma\in\Omega_n} |J_\sigma|^{s_*}\right)^p,$$

which implies

$$0 < \frac{1}{n} \log \left(\xi^{3s_*} \sum_{\sigma\in\Omega_n} |J_\sigma|^{s_*}\right) \text{ and so } \sum_{\sigma\in\Omega_n} |J_\sigma|^{s_*} > \xi^{-3s_*}.$$

Now if $h < s^*$, then $P(s^*) < 0$ as $P(t)$ is strictly decreasing. Then for any positive integer n, by Lemma 3.3, we have

$$0 > P(s^*) = \lim_{p\to\infty} \frac{1}{np} \log \sum_{\omega\in\Omega_{np}} |J_\omega|^{s^*} \geq \lim_{p\to\infty} \frac{1}{np} \log \xi^{-3(p-1)s^*} \left(\sum_{\sigma\in\Omega_n} |J_\sigma|^{s^*}\right)^p,$$

which implies

$$0 > \frac{1}{n} \log \left(\xi^{-3s^*} \sum_{\sigma\in\Omega_n} |J_\sigma|^{s^*}\right) \text{ and so } \sum_{\sigma\in\Omega_n} |J_\sigma|^{s^*} < \xi^{3s^*}.$$

Thus the proposition is obtained. □

Corollary 3.6. Since s_* and s^* be any two arbitrary real numbers with $0 < s_* < h < s^*$, from the above proposition it follows that for all $n \geq 1$,

$$\xi^{-3h} \leq \sum_{\sigma\in\Omega_n} |J_\sigma|^h \leq \xi^{3h}.$$

Proof. Let us first prove $\sum_{\sigma\in\Omega_n} |J_\sigma|^h \leq \xi^{3h}$. If not let $\sum_{\sigma\in\Omega_n} |J_\sigma|^h > \xi^{3h}$. Define $f_n(t) = \xi^{3t} - \sum_{\sigma\in\Omega_n} |J_\sigma|^t$. Then $f_n(t)$ is a real valued continuous function. Moreover, $f_n(s^*) > 0$, and $f_n(h) < 0$. Then by intermediate value theorem, there exists a real number s, where $h < s < s^*$, such that $f_n(s) = 0$, which implies $\sum_{\sigma\in\Omega_n} |J_\sigma|^s = \xi^{3s}$. But $h < s^*$ is arbitrary for which $\sum_{\sigma\in\Omega_n} |J_\sigma|^{s^*} < \xi^{3s^*}$, and so a contradiction arises. Hence $\sum_{\sigma\in\Omega_n} |J_\sigma|^h \leq \xi^{3h}$. Similarly, $\sum_{\sigma\in\Omega_n} |J_\sigma|^h \geq \xi^{-3h}$. Thus the corollary follows.

□

The following proposition plays an important role in the rest of the paper.

Proposition 3.7. Let $h \in (0, +\infty)$ be such that $P(h) = 0$. Then there exists a constant $\eta > 1$ and a probability measure μ_h supported by E such that for any $\sigma \in \Omega$,

$$\eta^{-1}|J_\sigma|^h \leq \mu_h(J_\sigma) \leq \eta |J_\sigma|^h.$$

Proof. For $\sigma \in \Omega$, $n \geq 1$, define

$$\nu_n(C(\sigma)) = \frac{\sum_{\tau\in\Omega_n} (\mathrm{diam} J_{\sigma\tau})^h}{\sum_{\tau\in\Omega_{|\sigma|+n}} (\mathrm{diam} J_\tau)^h}.$$

Then using Lemma 3.3 and Corollary 3.6, we have

$$\nu_n(C(\sigma)) \leq \frac{\xi^{3h}(\mathrm{diam} J_\sigma)^h \sum_{\tau\in\Omega_n} (\mathrm{diam} J_\tau)^h}{\sum_{\tau\in\Omega_{|\sigma|+n}} (\mathrm{diam} J_\tau)^h} \leq \xi^{9h}(\mathrm{diam} J_\sigma)^h,$$

and similarly, $\nu_n(C(\sigma)) \geq \xi^{-9h}(\mathrm{diam} J_\sigma)^h$. Thus for a given $\sigma \in \Omega$, $\{\nu_n(C(\sigma))\}_{n=1}^\infty$ is a bounded sequence of real numbers, and so Banach limit, denoted by Lim, is defined. For $\sigma \in \Omega$, let

$$\nu(C(\sigma)) = \mathrm{Lim}_{n\to\infty}\nu_n(C(\sigma)).$$

Then

$$\sum_{j=1}^N \nu(C(\sigma j)) = \mathrm{Lim}_{n\to\infty} \sum_{j=1}^N \frac{\sum_{\tau\in\Omega_n} (\mathrm{diam} J_{\sigma j\tau})^h}{\sum_{\tau\in\Omega_{|\sigma|+1+n}} (\mathrm{diam} J_\tau)^h} = \mathrm{Lim}_{n\to\infty} \frac{\sum_{\tau\in\Omega_{n+1}} (\mathrm{diam} J_{\sigma\tau})^h}{\sum_{\tau\in\Omega_{|\sigma|+n+1}} (\mathrm{diam} J_\tau)^h},$$

and so

$$\sum_{j=1}^N \nu(C(\sigma j)) = \mathrm{Lim}_{n\to\infty}\nu_{n+1}(C(\sigma)) = \mathrm{Lim}_{n\to\infty}\nu_n(C(\sigma)) = \nu(C(\sigma)).$$

Thus by Kolmogorov's extension theorem, ν can be extended to a unique Borel probability measure γ on Ω_∞. Let μ_h be the image measure of γ under the coding map π, i.e., $\mu_h = \gamma\circ\pi^{-1}$. Then μ_h is a unique Borel probability measure supported by E. Moreover, for any $\sigma \in \Omega$,

$$\mu_h(J_\sigma) = \gamma(C(\sigma)) = \mathrm{Lim}_{n\to\infty}\nu_n(C(\sigma)) \leq \mathrm{Lim}_{n\to\infty}\xi^{9h}(\mathrm{diam} J_\sigma)^h = \xi^{9h}(\mathrm{diam} J_\sigma)^h,$$

and similarly,

$$\mu_h(J_\sigma) \geq \xi^{-9h}(\mathrm{diam} J_\sigma)^h.$$

Write $\eta = \xi^{9h}$, and then $\eta > 1$, and thus the proof of the proposition is complete. □

The following proposition is useful.

Proposition 3.8. Let $r > 0$, and let $\mathcal{U}_r$ be the r-Moran covering of E. Then there exists a positive integer M such that the ball $B(x, r)$ of radius r, where $x \in J$, intersects at most M elements of $\mathcal{U}_r$.

Proof. By Proposition 2.5, for any $\sigma \in \Omega$, we get $|J_\sigma| \geq \xi^{-1}B^{-1}|J_{\sigma^-}|$. Let $\mathcal{U}_r$ be the r-Moran covering of E. Fix any $x \in J$, and write $V = B(x, r)$. Define,

$$Q_V = \{J_\sigma \in \mathcal{U}_r : J_\sigma \cap V \neq \emptyset,\ \sigma \in \Omega\}.$$

Note that any interval J_σ in Q_V contains a ball of radius $\frac{1}{2}|J_\sigma|$, and all such balls are disjoint. Moreover, all the balls are contained in a ball of radius $2r$ concentric with V, and so comparing the volumes (in fact lengths),

$$2r \geq (\#Q_V)\,\frac{1}{2}|J_\sigma| \geq (\#Q_V)\,\frac{1}{2}\xi^{-1}B^{-1}|J_{\sigma^-}| > (\#Q_V)\,\frac{1}{2}\xi^{-1}B^{-1}r,$$

which implies $\#Q_V < 4\xi B$. Hence $M := \lfloor 4\xi B\rfloor$ fulfills the statement of the proposition.

□

Proposition 3.9. Let $h \in (0, +\infty)$ be such that $P(h) = 0$. Then,

$$0 < \mathcal{H}^h(E) < \infty \text{ and } \dim_H(E) = h.$$

Proof. For any $n \geq 1$, the set $\{J_\sigma : \sigma \in \Omega_n\}$ is a covering of the E and so by Corollary 3.6,

$$\mathcal{H}^h(E) \leq \liminf_{n\to\infty} \sum_{\sigma\in\Omega_n} |J_\sigma|^h \leq \xi^{3h} < \infty,$$

which yields $\dim_H(E) \leq h$.

Let μ_h be the probability measure defined in Proposition 3.7, and $k \geq 1$. Then we get $\mu_h(J_\sigma) \leq \eta |J_\sigma|^h$. Let $r > 0$ and let $\mathcal{U}_r = \{J_\omega : |J_\omega| \leq r < |J_{\omega^-}|\}$ be the r-Moran covering of E. Then by Proposition 3.8, we get

$$\mu_h(B(x,r)) \leq \sum_{J_\omega \cap B(x,r)\neq\emptyset} \mu_h(J_\omega) \leq \eta \sum_{J_\omega \cap B(x,r)\neq\emptyset} |J_\omega|^h \leq \eta M r^h.$$

Thus

$$\limsup_{r\to 0} \frac{\mu_h(B(x,r))}{r^h} \leq \eta M,$$

and so by Proposition 2.2 in [F2], $\mathcal{H}^h(E) \geq \eta^{-1}M^{-1} > 0$, which implies $\dim_H(E) \geq h$. Thus the proposition is yielded. $\square$

Let us now prove the following lemma.

Lemma 3.10. *Let $h \in (0, +\infty)$ be such that $P(h) = 0$. Then $\overline{dim}_B(E) \leq h$.*

Proof. Let μ_h be the probability measure defined in Proposition 3.7, and for $r > 0$ let $\mathcal{U}_r = \{J_\omega : |J_\omega| \leq r < |J_{\omega^-}|\}$ be the r-Moran covering of E. Then for any $J_\sigma \in \mathcal{U}_r$, we get $\mu_h(J_\sigma) \geq \eta^{-1}|J_\sigma|^h$. Thus it follows that

$$\|\mathcal{U}_r\|^h = \sum_{J_\sigma\in\mathcal{U}_r} |J_\sigma|^h \leq \eta \sum_{J_\sigma\in\mathcal{U}_r} \mu_h(J_\sigma) = \eta.$$

Again for any $J_\sigma \in \mathcal{U}_r$, it follows that $|J_\sigma| \geq \xi^{-1}B^{-1}|J_{\sigma^-}| > \xi^{-1}B^{-1}r$. Hence, $(\xi B)^{-h} r^h N_r(E) \leq \|\mathcal{U}_r\|^h \leq \eta$, where $N_r(E)$ is the smallest number of sets of diameter at most r that can cover E, which implies $N_r(E) \leq \eta\,(\xi B)^h r^{-h}$ and so

$$\log N_r(E) \leq \log\left[\eta\,(\xi B)^h\right] - h\log r,$$

which yields

$$\overline{\dim}_B E = \limsup_{r\to 0} \frac{\log N_r(E)}{-\log r} \leq h,$$

and thus the lemma is obtained. $\square$

Let us now prove the following proposition.

Proposition 3.11. Let $h \in (0, +\infty)$ be such that $P(h) = 0$, and then $\mathcal{P}^h(E) < \infty$.

Proof. For the probability measure μ_h, by Proposition 3.7, there exists a constant $\eta \geq 1$ such that $\mu_h(J_\sigma) \geq \eta^{-1}|J_\sigma|^h$. Again Proposition 2.5 gives that

$$|J_\sigma| \geq \xi^{-1}B^{-1}|J_{\sigma^-}|.$$

Let $\mathcal{U}_r = \{J_\omega : |J_\omega| \leq r < |J_{\omega^-}|\}$ be the r-Moran covering of E for some $r > 0$. Let $x \in J_\sigma$ for some $J_\sigma \in \mathcal{U}_r$, and then $J_\sigma \subset B(x,r)$. Therefore,

$$\mu_h(B(x,r)) \geq \mu_h(J_\sigma) \geq \eta^{-1}|J_\sigma|^h > \eta^{-1}\xi^{-h}B^{-h}r^h,$$

which implies

$$\liminf_{r\to 0} \frac{\mu_h(B(x,r))}{r^h} \geq \eta^{-1}\xi^{-h}B^{-h},$$

and so by Proposition 2.2 in [F2],

$$\mathcal{P}^h(E) \leq 2^h \eta \xi^h B^h < \infty,$$

and thus the proposition is obtained. □

Proof of Theorem 3.1. Proposition 3.9 tells us that $\dim_H(E) = h$, and Lemma 3.10 gives that $\overline{\dim}_B E \leq h$. Combining these with the inequalities in (2), we have

$$\dim_H(E) = \dim_P(E) = \underline{\dim}_B(E) = \overline{\dim}_B(E) = h.$$

Again from Proposition 3.9 and Proposition 3.11 it follows that

$$0 < \mathcal{H}^h(E) \leq \mathcal{P}^h(E) < \infty.$$

Thus the proof of the theorem is complete.

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